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A qualitative analysis of the steady modes of operation of small wind power generators☆

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ABSTRACT

The motion of an electromechanical system that simulates the steady modes of operation of a generator driven by a wind turbine is investigated by qualitative methods of theoretical mechanics. A comparatively simple mathematical model that enables the influence of inertial, geometric, aerodynamic and electrodynamic characteristics of the system to be taken into account is studied. An approximation formula for the aerodynamic torque acting on the blades of the wind turbine, which is applicable in the region of high-speed steady modes of operation, is proposed. A theoretical and numerical investigation of high-speed steady modes is carried out using experimental data for the aerodynamic characteristics of standard blade foils. A description of the operating modes of a small wind power system and of their evolution, bifurcations and stability is given. Estimates of the characteristics of the optimal modes of operation are obtained. ©2009.

1. Model of the system

Consider a mechanical system consisting of a wind turbine and an electric generator. We will write the equation of motion of this system,[1–3] assuming that air flow has a constant velocity V and acts only on the blades:

$$J\dot{\omega} = M - cI, \quad L\dot{I} = c\omega - (R + r)I \tag{1.1}$$

Here ω is the angular velocity of the turbine, J is the moment of inertia of the turbine, M is the aerodynamic torque, I is the current strength in the armature winding, c is the electromechanical coupling factor, L and r are the inductance and internal resistance of the armature, and R is the external resistance.

The quantity cI represents the torque on the axis of the armature produced by the electromagnetic forces, and $c\omega$ is the electromotive force of induction of the armature.

For the aerodynamic torque, we use the representation[2,4]

$$M = \frac{1}{2}\rho S b V_a^2 (C_y(\alpha)\cos(\varphi - \alpha) - C_x(\alpha)\sin(\varphi - \alpha)) \tag{1.2}$$

Here ρ is the air density, S is the area of the blades, $C_x(\alpha)$ and $C_y(\alpha)$ are the drag and list coefficients of an individual blade, b is the distance from the effective centre of pressure of the blades to the axis of rotation, and φ is the effective pitch of the blades. The instantaneous angle of attack α and the air speed V_a of the effective centre of pressure are defined by the following relations

$$\alpha = \varphi - \operatorname{arctg}\frac{b\omega}{V}, \quad V_a^2 = (b\omega)^2 + V^2 \tag{1.3}$$

This model contains numerous design parameters, particularly b, φ, c, r, J and L.

It has been shown[1,2] that the function $M(\omega, \varphi)$ has the following property: for each φ in the range $\omega > 0$ there is a value $\omega_m(\varphi)$ of the angular velocity at which the torque reaches the maximum value $M_{\max}$.

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It should be noted that when $\varphi \sim 80°–90°$ relation (1.2) has a characteristic property: at comparatively low angular velocities the torque is significantly less than M_{max} and can even be negative (the torque may change from accelerating to decelerating).

2. Approximation of the aerodynamic torque

The aerodynamic torque is given by the complicated non-linear function (1.2), (1.3) of the angular velocity. A wind wheel that has blades with a high aerodynamic quality also can have high rotational velocity, and the relation $\omega b/V \gg 1$ holds for its preferential operating modes and can be used to derive a simple calculation formula for the aerodynamic torque. We will illustrate this possibility for the case when the pitch is close to $\pi/2$. Putting $\varphi = \pi/2 - \psi$, $\psi \ll 1$ and expanding the expression for the torque in $1/\omega$ to second-order terms we obtain

$$M_a = A_2\omega^2 + \psi A_1\omega + A_0 + \psi\frac{B_1}{\omega} + \frac{B_2}{\omega^2}$$

$$A_2 = \frac{1}{2}\rho S b^3\left(-C_{x0} - \frac{1}{2}C''_{x0}\psi^2\right) + o(\psi^2)$$

$$A_1 = \frac{1}{2}\rho S b^2 V\left(-C'_{y0} + C''_{x0} + \left[-\frac{1}{6}C'''_{y0} + \frac{1}{6}C''''_{x0}\right]\psi^2\right) + o(\psi^2)$$

$$A_0 = \frac{1}{2}\rho S b^2 V\left(C'_{y0} - \frac{1}{2}C_{x0} - \frac{1}{2}C''_{x0} + \left[\frac{1}{2}C'''_{y0} - \frac{1}{4}C''_{x0} - \frac{1}{4}C''''_{x0}\right]\psi^2\right) + o(\psi^2)$$

$$B_1 = \frac{1}{2}\rho S b V^2\left(-\frac{1}{2}C'_{y0} - \frac{1}{2}C'''_{y0} + \frac{1}{6}C''_{x0} + \frac{1}{6}C''''_{x0}\right) + O(\psi^2)$$

$$B_2 = \frac{1}{2}\rho S b^{-1} V^4\left(\frac{1}{6}C'_{y0} + \frac{1}{6}C'''_{y0} + \frac{1}{8}C_{x0} + \frac{1}{12}C''_{x0} - \frac{1}{24}C''''_{x0}\right) + O(\psi^2) \tag{2.1}$$

A subscript zero indicates that the values of the functions $C_x(\alpha)$ and $C_y(\alpha)$ and their derivatives were taken at $\alpha = 0$. Here we utilized the fact that for symmetrical foils C_x is an even function, and C_y is an odd function of the angle of attack α.

We also note that the following relations hold for foils with a relatively high quality

$$|C'''_{y0}| \gg 1 \gg C_{x0}, \quad |C'''_{y0}| \gg C''_{x0} \gg C_{x0} > 0 \tag{2.2}$$

$$|C'''_{y0}| \gg C'''_{x0} \gg 0, \quad |C'''_{y0}| \gg C'_{y0} \gg C_{x0}, \quad C'''_{y0} < 0 \tag{2.3}$$

Hence for sufficiently small values of ψ we obtain

$$A_2 < 0, \quad B_2 < 0, \quad B_1 > 0$$

The presence of the non-zero coefficients A_1, B_1 and B_2 in (2.1) distinguishes this formula from the formulae usually used in wind power engineering (see, for example, Ref. 5).

Using an approximation formula, we can estimate the value ω_m (the value ω_0) of the angular velocity at which the torque reaches a maximum (vanishes):

$$\omega_m = \bar{\omega}_m + \psi\frac{B_1\bar{\omega}_m^2 - A_1\bar{\omega}_m^5}{4B_2} + o(\psi), \quad \bar{\omega}_m = \left(\frac{B_2}{A_2}\right)^{1/4}$$

$$\omega_0 = \bar{\omega}_0 - \psi\frac{A_1\bar{\omega}_0^3 + B_1\bar{\omega}_0^2}{2A_2\bar{\omega}_0^4 + A_1\bar{\omega}_0^3 - 2B_2} + o(\psi) \quad \bar{\omega}_0 = \sqrt{\frac{A_0 + \sqrt{A_0^2 - 4A_2B_2}}{2A_2}} \tag{2.4}$$

3. Steady motions

The equations of the steady motions of the dynamical system under consideration have the form

$$I = M(\omega, \varphi)/c \tag{3.1}$$

$$I = c\omega/(R + r) \tag{3.2}$$

In Fig. 1 a graph of (3.1) is shown in the phase plane ($\Omega = b\omega/V, I$) for the pitch value $\psi = 0.1$. The experimental aerodynamic characteristics $C_x(\alpha)$ and $C_y(\alpha)$ (Ref. 6) of the standard NACA 0012 airfoil were used in the calculation. The small circles show the values calculated from these data using formula (1.2). The solid line depicts the approximation relation (2.1). Here we have

$$A_2 \approx -0.013, \quad A_1 \approx -0.54, \quad A_0 \approx 4.55, \quad B_1 \approx 19.13, \quad B_2 \approx -63.65$$

Straight lines given by (3.2) are shown for two values of the external resistance when $b = 0.2$ m, $S = 0.05$ m^2, $r = 10\,\Omega$, $c = 0.5$ V s, $V = 5$ m/s

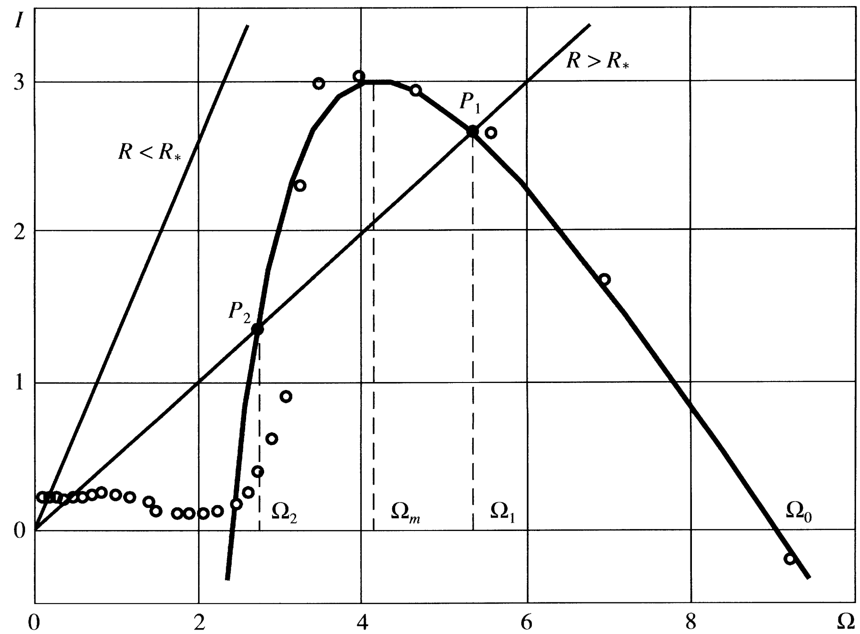

Fig. 1.

The points of intersection are fixed points of the system for the corresponding value of R. It can be seen that there is a range of values of R in which the system has several fixed points. This range can be fairly broad. The hysteresis previously described in Ref. 1, which can apparently disturb the normal functioning of the system, is attributed specifically to this phenomenon.

We will henceforth consider the range of relatively high angular velocities (which are most preferable from the practical point of view). This enables us to use approximation formula (2.1) for the aerodynamic torque. In this range there are no more than two fixed points (Fig. 1). We will use $P_1(\omega_1, I_1)$ to denote the fixed point that corresponds to the higher angular velocity, and we will use $P_2(\omega_2, I_2)$ to denote the other fixed point.

We will trace the evolution and bifurcations of the fixed points P_1 and P_2 as the parameters R and L vary.

When an additional consumer of electrical power is connected (as a rule, in parallel), the external resistance R decreases and may vanish (when short-circuit occurs). Accordingly, if the internal resistance r of the generator is sufficiently small, there is a value R_* of the external resistance for which the fixed point P_1 merges with P_2. We have

$$R_* = \frac{c^2\omega_*^3}{2A_2\omega_*^4 - 2B_2} - r > 0, \quad \omega_* = \sqrt{\frac{A_0 - \sqrt{A_0^2 + 12A_2B_2}}{2A_2}} + O(\psi) \tag{3.3}$$

Therefore, when $R < R_*$, the fixed points P_1 and P_2 do not exist, and there is no "preferable" operating mode.

It can be shown that the stability conditions have the form

$$\frac{R+r}{L} - \frac{M'_\omega(\omega_s, \varphi)}{J} > 0, \quad \frac{c^2}{R+r} - M'_\omega(\omega_s, \varphi) > 0 \tag{3.4}$$

Hence it follows that the fixed point P_2, as long as it exists, is a saddle point.

Consider the point P_1.

In the case in which $L \le L_* = J(R_* + r)^2 c^{-2}$, the point P_1 is asymptotically stable over the entire range of values of the external resistance $R > R_*$. If $L > L_*$, the nature of the stability depends on the relations between the parameters.

We will determine the value R_m of the resistance at which the fixed point P_1 corresponds to the maximum of the aerodynamic torque ($\omega_1 = \omega_m$) (Fig. 1):

$$R_m = c^2\omega_m(A_2\omega_m^2 + \psi A_1\omega_m + A_0 + \psi B_1\omega_m^{-1} + B_2\omega_m^{-2})^{-1} - r > R_*$$

If $L > L_*$ and $R_* < R \le R_m$, the stability conditions (3.4) for the point P_1 may be violated. In particular, if

$$L > L_1 = J(R+r)(\psi A_1 + 2A_2\omega - 2B_2\omega^{-3} - \psi B_1\omega^{-2})^{-1}$$

the first former of the conditions indicated is violated. When the value L_1 is superseded, an Andronov–Hopf bifurcation occurs. The nature of this bifurcation is specified[7] by the sign of the following expression

$$Q = 2(\psi A_1 + 2A_2\omega - 2B_2\omega^{-3} - \psi B_1\omega^{-2})(A_2 + 3B_2\omega^{-4} + \psi B_1\omega^{-3})^2 -$$
$$- 3J((\psi A_1 + 2A_2\omega - 2B_2\omega^{-3} - \psi B_1\omega^{-2})^2 J^{-1} - c^2L^{-1})(-4B_2\omega^{-5} - \psi B_1\omega^{-4})$$

Note that in the range of values of the external resistance considered,

$$M'_{\omega} = \psi A_1 + 2A_2\omega - 2B_2\omega^{-3} - \psi B_1\omega^{-2} > 0$$

and the second condition in (3.4) unquestionably holds for P_1. Taking relation (2.2) into account, we can show that $Q > 0$. This means that merging of an unstable cycle with a stable focus accompanied by a loss of stability by the latter (a hard loss of stability) occurs as L is increased.

A numerical analysis reveals that this unstable cycle emerges from the separatrix loop of the saddle point P_2.

In the range $R_m < R < \infty$, the fixed point P_1 is asymptotically stable for any values of the inductance. However, the nature of the decay in the vicinity of this fixed point depends on L: at sufficiently small and sufficiently large values of L, this fixed point is a node, and at intermediate values of L, it is a focus.

In the case of an open circuit ($R = \infty$), the point P_1 is asymptotically stable at any value of L and corresponds to rotation of the turbine with angular velocity ω_0. The current, of course, is equal to zero in this case.

It is also of interest to examine the influence of the free-stream velocity V on the steady modes of operation. It follows from equality (2.1) that $M \sim V^2$. In addition, $\omega_m = V\Omega_m$, and $\omega_* = V\Omega_*$, where Ω_* and Ω_m do not depend on V. At the same time, the "electrical" parameters (r, c and R) do not depend on V. Therefore, for any value of the external resistance, there is a minimum value of the wind velocity

$$V_* = \frac{c^2\Omega_*^3}{(R + r)(2A_2\Omega_*^4 - 2B)} \tag{3.5}$$

below which the fixed points P_1 and P_2 are absent. Here $B = B_2V^{-4}$ is a quantity that does not depend on V.

4. Optimization

One of the output characteristics of a wind-turbine is the torque on the shaft. Therefore, the quantity $M_{\max}$ may be of interest for a specific class of problems. Using the approximation formula (2.1), we obtain (setting, for simplicity, $\psi = 0$)

$$M_{\max} = A_0 - 2\sqrt{A_2B_2} \tag{4.1}$$

At the same time, in certain situations (for example, when a wind-turbine is used to drive a pump etc.) the mechanical power N "trapped" by the wind turbine ($N = \omega M$) is important. The problem of maximizing this quantity is discussed in the literature devoted to wind power systems. From equality (2.1) we have

$$N = A_2\omega^3 + \psi A_1\omega^2 + A_0\omega + \psi B_1 + \frac{B_2}{\omega} \tag{4.2}$$

The power reaches a maximum at the angular velocity ω_n. We estimate it for $\psi = 0$:

$$\omega_n = \frac{A_0 + \sqrt{A_0^2 + 12A_2B_2}}{-6A_2}$$

In principle, the direction in which the blades should be turned in order to ensure an increase in the torque or the mechanical power can be found using approximation formulae (2.1) and (2.4).

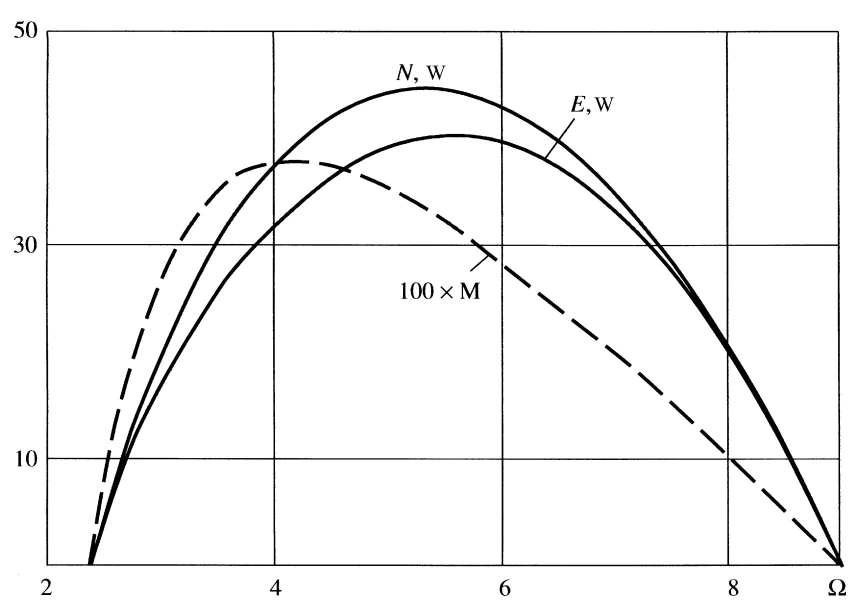

Fig. 2.

Note that the primary characteristic of a wind power system is the electrical power $E = RI^2$ produced by it. Owing to the fact that the previously described closed model[1,2] relates the electrical and mechanical characteristics of the system, it is now possible to represent this quantity at operating mode in the following form (taking Eqs (3.1) and (3.2) into account)

$$E = N - \frac{r}{c^2}M^2$$

In the general case, the maxima of the functions N and E are clearly reached at the different values ω_n and ω_e of the angular velocity, where $\omega_e > \omega_n$.[2]

Fig. 2 presents approximation graphs of $N(\Omega)$ and $E(\Omega)$ for the system parameters indicated. For comparison, the dashed curve depicts the function $M(\Omega)$ multiplied by a scale factor.

Despite the fact that the proposed model is relatively simple and "rough," it can be used to carry out an effective qualitative analysis of a system and thereby to narrow the region in which a more detailed investigation involving complex models that require large-scale numerical computations is required.

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